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Dimension Spectrum of Axiom A Diffeomorphisms. II. Gibbs Measures

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We compute the dimension spectrum $f(\alpha)$ of the singularity sets of a Gibbs measure defined on a two-dimensional compact manifold and invariant with respect to a C^2 Axiom A diffeomorphism. This case is the generalization of the case where the measure studied is the Bowen–Margulis measure—the one that realizes the topological entropy. We obtain similar results; for example, the function f is the Legendre–Fenchel transform of a free energy function which is real analytic (linear in the degenerate case). The function f is also real analytic on its definition domain (defined in one point in the degenerate case) and is related to the Hausdorff dimensions of Gibbs measures singular with respect to each other and whose supports are the singularity sets, and we finally decompose these sets.

KEY WORDS: Multifractal; thermodynamic formalism; Hausdorff dimension; free energy function; large deviations; Gibbs measures.

INTRODUCTION

Our aim is to study the dimension spectrum of a Gibbs measure defined on a two-compact manifold and invariant with respect to a C^2 Axiom A diffeomorphism. This article follows ref. 21, where we studied the same problem in an important particular case, the case where the measure is the Bowen–Margulis measure, the one that achieves the maximum of topological entropy (it is the Gibbs measure of 0); the present paper is the second part of this work. We shall see that the results are very similar, and the proofs parallel in general those in ref. 21. Let us remark that the original idea of this work comes from ref. 2, where one studies the dimension

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spectrum of some dynamical systems (Markovian transformations) defined on the interval or the circle.

Multifractal analysis is concerned with the study of ratios of the measures of small sets U ($|U| \to 0$, where $|U|$ denotes its diameter), and we want to obtain some information in the so-called singularity sets defined for any positive real α:

$$C_\alpha^+ = \{x/\alpha^+(x) = \alpha\}, \qquad C_\alpha^- = \{x/\alpha^-(x) = \alpha\}, \qquad C_\alpha = C_\alpha^+ \cap C_\alpha^- \tag{0.1}$$

where the maps α^+ and α^- denote local dimensions and are defined by

$$\alpha^+(x) = \overline{\lim_{\substack{x \in \mathrm{int}(U) \\ |U| \to 0}}} \frac{\mathrm{Log}\, \mu(U)}{\mathrm{Log}\, |U|} \qquad \text{and} \qquad \alpha^-(x) = \underline{\lim_{\substack{x \in \mathrm{int}(U) \\ |U| \to 0}}} \frac{\mathrm{Log}\, \mu(U)}{\mathrm{Log}\, |U|} \tag{0.2}$$

There exists a unique real α (when the measure μ is ergodic) which satisfies

$$\alpha(x) = \alpha \quad \mu \text{ a.e.} \qquad \text{and} \qquad \mu(C_\alpha) = 1$$

In the case where for different values of α the sets C_α are not empty, we compute their Hausdorff dimensions in order to differentiate them, and we define the dimension spectrum $f(\alpha)$ by

$$f(\alpha) = HD(C_\alpha) \qquad (\text{and } f \equiv -\infty \text{ if } C_\alpha = \emptyset) \tag{0.3}$$

Our main results are:

- $f(\alpha) = HD(C_\alpha)$ for $\alpha \in [\alpha_1; \alpha_2] \subset \mathbb{R}^{+*}$ and $f \equiv -\infty$ otherwise.
- f is real analytic on $]\alpha_1; \alpha_2[$ when $\alpha_1 \neq \alpha_2$ (in particular, we make explicit the degenerate case $\alpha_1 = \alpha_2$).
- $f(\alpha) = HD(\mu_\alpha) = \inf\{HD(A)/\mu_\alpha(A) = 1\}$, where μ_α is a measure whose support is C_α, and there exist positive reals τ and η such that

$$[C_\tau^u; C_\eta^s] \subset C_\alpha$$

where $[\cdot;\cdot]$ denotes a local product,[1,18,21] and these two sets have the same Hausdorff dimensions.

The notations and definitions are the same as in ref. 21. The measure μ is the Gibbs measure of a real Hölder continuous function $\phi: \Lambda \to \mathbb{R}$, where the basic set Λ is the support of the measure μ, which is the unique measure which achieves the pressure of ϕ,

$$P_\phi = P_g(\phi) = \sup_{\rho \in M_g(\Lambda)} \left[h_\rho + \int \phi \, d\rho \right]$$

Analogy with Part I

We first establish a "decomposition" of the measure μ, not like a product such as the Bowen–Margulis measure, but like

$$c \leqslant \frac{d\mu}{d(\mu^u \times \mu^s)} \leqslant C$$

where c and C are positive constants and μ^u (resp. μ^s) is a measure defined on the unstable (resp. stable) manifold.

With this decomposition we prove that f is the Legendre–Fenchel transform of a free energy function F, i.e.,

$$f(\alpha) = \inf_{t \in \mathbb{R}} \{t\alpha - F(t)\} \tag{0.4}$$

where F decomposes into $F^u + F^s$ (unstable and stable free energy functions); the function F is obtained from a partition function defined for any real β by

$$Z_n(\beta) = \sum_{\substack{U \in U_n \\ \mu(U) > 0}} \mu(U)^\beta$$

where U_n is a partition whose diameter goes to 0 when n goes to $+\infty$, and we have for any real β

$$F(\beta) = \lim_{n \to +\infty} -\frac{1}{n} \operatorname{Log} Z_n(\beta) \tag{0.5}$$

We relate then f with other functions f^u and f^s (the Legendre–Fenchel transforms of F^u and F^s, and by the way the dimension spectra of the measures μ^u and μ^s), and also to the Hausdorff dimensions of Gibbs measures whose supports are the singularity sets of the measures μ^u and μ^s.

1. DECOMPOSITION OF THE MEASURE μ

We associate to the measure μ a measure ν defined on the subshift of finite type Σ_A

$$\mu = \pi^* \nu$$

and the measure ν is the Gibbs measure of the real Hölder continuous function $\zeta: \Sigma_A \to \mathbb{R}$, where we have[1]

$$\zeta = \phi \circ \pi \qquad \text{and} \qquad P_\zeta = P_\sigma(\zeta) = P_\phi$$

Let us define the Jacobian of the measure ν on Σ_A^+ by

$$\operatorname{Jac} \nu(\underline{x}) = \lim_{\substack{\underline{x} \in \operatorname{int}(C) \\ |C| \to 0}} \frac{\nu(\sigma(C))}{\nu(C)} \tag{1.1}$$

where the sets C are cylinders. This Jacobian is also the Radon–Nikodyn derivative of the measure ν over Σ_A^+, which is ν a.e. equal to a Hölder continuous function. Let us denote

$$\zeta^u = -\operatorname{Log} \operatorname{Jac} \nu \tag{1.2}$$

Let us remark that the function ζ^u is Hölder continuous and that $\zeta^u < 0$. We then define its Gibbs measure μ_{ζ^u}.

We associate to the function ζ^u the Ruelle–Perron–Froebenius operator on $C(\Sigma_A^+) : L$.[13] For any function $K \in C(\Sigma_A^+)$, we compute $L(K)$ applied to any $\underline{x} \in \Sigma_A^+$ with the formula

$$[L(K)](\underline{x}) = \sum_{\substack{\underline{y}: \\ \sigma(\underline{y}) = \underline{x}}} K(\underline{y}) \exp\{\zeta^u(\underline{y})\}$$

and the iterations for any integer n

$$[L^n(K)](\underline{x}) = \sum_{\substack{\underline{y}: \\ \sigma^n(\underline{y}) = \underline{x}}} K(\underline{y}) \exp\left\{\sum_{j=0}^{n-1} \zeta^u[\sigma^j(\underline{y})]\right\} \tag{1.3}$$

We have the following results[18]:

$$\lim_{n \to +\infty} \frac{1}{n} \operatorname{Log}[L^n(1)](\underline{x}) = P(\zeta^u) \qquad \text{for any} \quad \underline{x} \in \Sigma_A^+$$

and:

- There exists a Hölder continuous function $h^+ \in C(\Sigma_A^+)$ such that

$$h^+ > 0 \qquad \text{and} \qquad L(h^+) = h^+$$

- There exists a σ-invariant measure $\rho^+ \in M_\sigma(\Sigma_A^+)$ such that

$$L^*(\rho^+) = \rho^+$$

- The measure $\nu^+ = h^+ \rho^+ \in M_\sigma(\Sigma_A^+)$.

We have then

$$\nu^+ = \mu_{\zeta^u} \qquad \text{on} \quad \mathscr{B}(\Sigma_A^+)$$

where $\mathscr{B}(\Sigma_A^+)$ denotes the Borel algebra of Σ_A^+, and the measure ν^+ satisfies[12]

$$h_{\nu^+}(\sigma)+\int \zeta^u \, d\nu^+ = 0$$

which implies $P(\zeta^u)=0$ and for any measure $\rho \in M_\sigma(\Sigma_A^+)$, $\rho \neq \nu^+$, we have[8, 12]

$$h_\rho(\sigma)+\int \zeta^u \, d\rho < 0$$

By the same method we introduce the function ζ^s on Σ_A^- by

$$\zeta^s = -\mathrm{Log}\,\mathrm{Jac}\,\nu \tag{1.4}$$

where we have

$$\mathrm{Jac}\,\nu(\underline{x}) = \lim_{\substack{x \in \mathrm{int}(C) \\ |C| \to 0}} \frac{\nu(\sigma^{-1}(C))}{\nu(C)}$$

The function ζ^s is Hölder continuous and $\zeta^s<0$, and we have:

- There exists a Hölder continuous function $h^- \in C(\Sigma_A^-)$ such that

$$h^- > 0 \qquad \text{and} \qquad L(h^-)=h^-$$

- There exists a σ-invariant measure $\rho^- \in M_\sigma(\Sigma_A^-)$ such that

$$L^*(\rho^-)=\rho^-$$

- The measure $\nu^- = h^-\rho^- \in M_\sigma(\Sigma_A^-)$.

We have then

$$\nu^- = \mu_{\zeta^s} \qquad \text{on} \qquad \mathscr{B}(\Sigma_A^-)$$

where μ_{ζ^s} is the Gibbs measure of the function ζ^s. Moreover, the pressure $P(\zeta^s)=0$ and it is only achieved by the measure ν^-.

There exist positive constants c_i and C_i which bound the ratios in the definition of the Gibbs measures:

$$c_1 \leqslant \frac{\nu^+\{\underline{y} \in \Sigma_A^+ / x_0 = y_0, ..., x_p = y_p\}}{\exp\{\sum_{j=0}^{p} \zeta^u[\sigma^j(\underline{x})]\}} \leqslant C_1 \tag{1.5}$$

since we have $\nu^+ = \mu_{\zeta^u}$ on $\mathscr{B}(\Sigma_A^+)$, and

$$c_2 \leqslant \frac{\nu^-\{\underline{y} \in \Sigma_A^- / x_{-k} = y_{-k},..., x_0 = y_0\}}{\exp\{\sum_{j=-k}^{0} \zeta^s[\sigma^j(\underline{x})]\}} \leqslant C_2 \tag{1.6}$$

since we have $\nu^- = \mu_{\zeta^s}$ on $\mathscr{B}(\Sigma_A^-)$.

The expressions (1.5) and (1.6) imply that there exist positive constants c and C such that

$$c \leqslant \frac{\nu\{\underline{y} \in \Sigma_A / x_{-k} = y_{-k},..., x_p = y_p\}}{\nu^+\{\underline{y} \in \Sigma_A^+ / x_0 = y_0,..., x_p = y_p\} \times \nu^-\{\underline{y} \in \Sigma_A^- / x_{-k} = y_{-k},..., x_0 = y_0\}} \leqslant C$$

We have thus obtained

$$c \leqslant \frac{d\nu}{d(\nu^+ \times \nu^-)} \leqslant C \tag{1.7}$$

If we associate to the measure ν^+ (respectively ν^-) the measure μ^u (resp. μ^s) defined on the unstable (resp. stable) manifolds, we verify that[19]

$$c \leqslant \frac{d\mu}{d(\mu^u \times \mu^s)} \leqslant C \tag{1.8}$$

Let us denote the functions ξ^u and ξ^s defined on Λ and which satisfy

$$\zeta^u = \xi^u \circ \pi \qquad \text{and} \qquad \zeta^s = \xi^s \circ \pi \tag{1.9}$$

which means that for $x = \pi(\underline{x})$, we have, for instance, $\xi^u(x) = \zeta^u(\underline{x})$, and the measure μ^u (resp. μ^s) is the Gibbs measure of the Hölder continuous function ξ^u (resp. ξ^s). (Remember that in ref. 21 we have $\xi^u = \xi^s = -h$.)

We use this decomposition to compute the free energy functions F^u and F^s associated with partitions situated on unstable and stable manifolds and to the measures μ^u and μ^s, and then the free energy function F associated with the local product of the previous partitions and to the measure μ.

2. EXISTENCE AND REGULARITY OF THE FREE ENERGY FUNCTION

Let us define the partitions $(U_n^u)_{n \geqslant 1}$ and $(U_n^s)_{n \geqslant 1}$ defined on the unstable and stable manifolds. Denote $U_n = [U_n^u, U_n^s]$ the local product partition defined on the basic set Λ and which is composed of sets $U = [U_1; U_2]$, where we have $U_n = [U_n^u; U_n^s]$.[18, 21]

We shall use this partition in order to decompose the associated free energy function F into the sum $F^u + F^s$.

2.1. Decomposition of the Free Energy Function

Using (1.8), we get for any $U = [U_1; U_2]$

$$c\mu^u(U_1)\,\mu^s(U_2) \leqslant \mu(U) \leqslant C\mu^u(U_1)\,\mu^s(U_2) \tag{2.1.1}$$

and we have thus

$$\mu(U) > 0 \Leftrightarrow \mu^u(U_1)\,\mu^s(U_2) > 0$$

It follows that we have for any real β

$$\inf(c^\beta; C^\beta)\,\mu^u(U_1)^\beta\,\mu^s(U_2)^\beta \leqslant \mu(U)^\beta \leqslant \sup(c^\beta; C^\beta)\,\mu^u(U_1)^\beta\,\mu^s(U_2)^\beta \tag{2.1.2}$$

If we define as in (0.5)

$$F_n^u(\beta) = -\frac{1}{n}\,\mathrm{Log} \sum_{\substack{U_1 \in U_n^u \\ \mu^u(U_1) > 0}} \mu^u(U_1)^\beta$$

$$F_n^s(\beta) = -\frac{1}{n}\,\mathrm{Log} \sum_{\substack{U_2 \in U_n^s \\ \mu^s(U_2) > 0}} \mu^s(U_2)^\beta$$

$$F_n(\beta) = -\frac{1}{n}\,\mathrm{Log} \sum_{\substack{U \in U_n \\ \mu(U) > 0}} \mu(U)^\beta$$

we obtain for any real β

$$|F_n(\beta) - F_n^u(\beta) - F_n^s(\beta)| \leqslant \frac{|\beta|}{n} \sup(|\mathrm{Log}\, c|; |\mathrm{Log}\, C|) \tag{2.1.3}$$

and this implies that the sequence of functions $(F_n)_{n \geqslant 1}$ is convergent if and only if the sequences $(F_n^u)_{n \geqslant 1}$ and $(F_n^s)_{n \geqslant 1}$ are also convergent; we get therefore in that case for any real β

$$F(\beta) = \lim_{n \to +\infty} F_n(\beta) = \lim_{n \to +\infty} F_n^u(\beta) + \lim_{n \to +\infty} F_n^s(\beta) = F^u(\beta) + F^s(\beta) \tag{2.1.4}$$

It suffices therefore to compute the functions F^u and F^s in order to get F.

2.2. Computation of the Free Energy Function

The unstable free energy function is given by the following result.

Theorem 2.2.1. We have for any real β

$$F^u(\beta) = \inf_{\rho \in M_g(A)} \left[\frac{h_\rho + \beta \int \xi^u \, d\rho}{\int J^u \, d\rho} \right]$$

Remarks. This functional is a large-deviations functional and it is lower semicontinuous.

It is achieved by a unique measure μ^u_β, which is the Gibbs measure of the Hölder continuous function $\beta\xi^u - F^u(\beta)\, J^u$.

The proof parallels the proof of Theorem 2.2.1 in ref. 21. It is divided into three steps:

Proposition 2.2.2. We have for any real β

$$\underline{\lim}_{n \to +\infty} -F^u_n(\beta) \geqslant \sup_{\substack{\rho \in M_g(A) \\ \rho \text{ ergodic}}} \left[\frac{h_\rho + \beta \int \xi^u\, d\rho}{\int -J^u\, d\rho}\right]$$

Proposition 2.2.3. We have for any real β

$$\sup_{\substack{\rho \in M_g(A) \\ \rho \text{ ergodic}}} \left[\frac{h_\rho + \beta \int \xi^u\, d\rho}{\int -J^u\, d\rho}\right] = \sup_{\rho \in M_g(A)} \left[\frac{h_\rho + \beta \int \xi^u\, d\rho}{\int -J^u\, d\rho}\right]$$

Proposition 2.2.4. We have for any real β

$$\overline{\lim}_{n \to +\infty} -F^u_n(\beta) \leqslant \sup_{\rho \in M_g(A)} \left[\frac{h_\rho + \beta \int \xi^u\, d\rho}{\int -J^u\, d\rho}\right]$$

We give here the sketch of the proof of Proposition 2.2.4. We take the partitions $(U^u_n)_{n \geqslant 1}$ and $(U^s_n)_{n \geqslant 1}$ to be uniform: $|U| \simeq e^{-n}$ ($\simeq$ signifies that the ratios of the two members are bounded by constants). Following refs. 18 and 21, we associate to any interval $U \in U^u_n$ an integer $n(U)$ (which we can identify as the size of the interval—or the associated cylinder C—and e^{-n} represents its length) and an element $y(U) \in U$ (the center of the cylinder) such that:

- The cylinder $C = \{\underline{x} \in \Sigma^+_A / (\underline{x})_i = (\underline{y}(U))_i,\ 0 \leqslant i < n(U)\}$ satisfies

$$\mu^u(\pi(C)\, \Delta U) = 0 \quad [\text{or } \mu^u(U) \simeq \nu^+(C)]$$

- $|g^n(U)| \simeq 1$.

We have then

$$\exp\left\{\sum_{j=0}^{n(U)-1} J^u[g^j(y(U))]\right\} \simeq |U| \simeq e^{-n} \tag{2.2.1}$$

and

$$\exp\left\{\sum_{j=0}^{n(U)-1} \xi^u[g^j(y(U))]\right\} \simeq \mu^u(U) \tag{2.2.2}$$

We put together the intervals U corresponding to the same "size"; let then

$$E_i = \{U \in U_n^u / n(U) = i\} \tag{2.2.3}$$

The sets E_i are only defined for integers i varying in a linear scale, since, using (2.2.1), we get

$$i \in \left[\frac{n}{\sup - J^u}; \frac{n}{\inf - J^u}\right] = [na_1; na_2]$$

At the rank n there exists for any real β an integer $i(n)$ $[=i(n, \beta)]$ such that

$$\sum_{U \in E_i} \mu^u(U)^\beta \leqslant \sum_{U \in E_{i(n)}} \mu^u(U)^\beta$$

and then we obtain

$$-F_n^u(\beta) \sim \frac{1}{n} \operatorname{Log}\left\{\sum_{U \in E_{i(n)}} \mu^u(U)^\beta\right\} \tag{2.2.4}$$

We define for integers k the sets

$$K_k = \left\{U \in E_{i(n)} \middle/ -\sum_{j=0}^{n(U)-1} \xi^u[g^j(y(U))] \in [k; k+1[\right\} \tag{2.2.5}$$

The sets K_k are only defined for integers k varying in a linear scale, since, using (2.2.1) and (2.2.2), we get

$$k \in [i(n) \inf -\xi^u; i(n) \sup -\xi^u] = [cn; dn]$$

This implies the existence of an integer $k(n)$ $[=k(n, \beta)]$ such that

$$\sum_{U \in K_k} \mu^u(U)^\beta \leqslant \sum_{U \in K_{k(n)}} \mu^u(U)^\beta$$

and we get therefore

$$-F_n^u(\beta) \sim \frac{1}{n} \operatorname{Log}\left\{\sum_{U \in K_{k(n)}} \mu^u(U)^\beta\right\} \tag{2.2.6}$$

Let us remark that the intervals $U \in K_{k(n)}$ have the same size $i(n)$ and μ measure $\exp\{-k(n)\}$. It follows from (2.2.6) that we have

$$-F_n^u(\beta) \sim \frac{1}{n} \operatorname{Log} \# K_{k(n)} - \beta \frac{k(n)}{n}$$

or also

$$-F_n^u(\beta) \sim \frac{1}{n} \operatorname{Log} \# K_{k(n)} - \beta \frac{i(n)}{n} \frac{k(n)}{i(n)} \tag{2.2.7}$$

Let us define the probability measures

$$\theta_n = \frac{1}{\# K_{k(n)}} \sum_{U \in K_{k(n)}} \delta_{y(U)} \qquad \text{and} \qquad \rho_n = \frac{1}{i(n)} \sum_{j=0}^{i(n)-1} g^j \theta_n \tag{2.2.8}$$

The sequences

$$\frac{1}{n} \operatorname{Log} \# K_{k(n)} \qquad \text{and} \qquad \rho_n$$

take their values in compact sets $[0; 1]$ and $M(\Lambda)$. We can suppose that they converge (there exists a common subsequence ...) and we get

$$\frac{1}{n} \operatorname{Log} \# K_{k(n)} \to \gamma \in [0; 1] \text{ and } \rho_n \to \rho \in M_g(\Lambda) \quad \text{(the limit is } g\text{-invariant!)}$$

Let us compute the integral

$$\int J^u \, d\rho_n = \frac{1}{\# K_{k(n)}} \sum_{U \in K_{k(n)}} \left\{ \frac{1}{i(n)} \sum_{j=0}^{i(n)-1} J^u[g^j(y(U))] \right\}$$

Using (2.2.1), (2.2.3), and (2.2.5), we have for any $U \in K_{k(n)}$

$$\frac{1}{i(n)} \sum_{j=0}^{i(n)-1} J^u[g^j(y(U))] \sim \frac{-n}{i(n)}$$

and this implies

$$\int J^u \, d\rho_n \sim \frac{-n}{i(n)}$$

We get therefore

$$\lim_{n \to +\infty} \frac{i(n)}{n} = \gamma' = \frac{1}{\int -J^u \, d\rho} \tag{2.2.9}$$

We obtain similarly

$$\lim_{n \to +\infty} \frac{k(n)}{i(n)} = \gamma'' = \int -\xi^u \, d\rho \tag{2.2.10}$$

or also

$$\lim_{n \to +\infty} \frac{k(n)}{n} = \frac{\int -\xi^u \, d\rho}{\int -J^u \, d\rho}$$

An upper bound of the real γ is given by

$$\gamma \leqslant \gamma' h_\rho \leqslant \frac{h_\rho}{\int -J^u \, d\rho} \tag{2.2.11}$$

using a proof due to Misiurewicz (ref. 4, p. 145) adapted in ref. 21.

We obtain therefore

$$\lim_{n \to +\infty} -F_n^u(\beta) = \gamma - \beta\gamma'\gamma'' \leqslant \frac{h_\rho + \beta \int \xi^u \, d\rho}{\int -J^u \, d\rho}$$

(remember that this is a subsequence). This result implies Proposition 2.2.4.

By the same method we prove the following (replace g by g^{-1})

Theorem 2.2.5. We have for any real β

$$F^s(\beta) = \inf_{\rho \in M_g(\Lambda)} \left[\frac{h_\rho + \beta \int \xi^s \, d\rho}{\int J^s \, d\rho} \right]$$

2.3. Regularity of the Free Energy Function and of Its Legendre–Fenchel Transform

Let us define the functions G^u and G^s for any pair $(x, y) \in \mathbb{R}^2$ by

$$G^u(x, y) = \sup_{\rho \in M_g(\Lambda)} \left[h_\rho + \int (x\xi^u + yJ^u) \, d\rho \right] = P(x\xi^u + yJ^u)$$

and

$$G^s(x, y) = P(x\xi^s + yJ^s)$$

which represent the pressures of the Hölder continuous functions $x\xi^u + yJ^u$ and $x\xi^s + yJ^s$ (and are dynamical free energy functions). We know that they are real analytic in both variables[18] and it is easy to prove that we have for any real β

$$G^u(\beta, -F^u(\beta)) = G^s(\beta, -F^s(\beta)) = 0 \tag{2.3.1}$$

If we denote by μ^u_β (respectively μ^s_β) the Gibbs measure of the function $\beta\xi^u - F^u(\beta) J^u$ [resp. $\beta\xi^s - F^s(\beta) J^s$] we prove that[14, 15, 18, 21]

$$\left(\frac{\partial G^u}{\partial y}\right)(\beta, -F^u(\beta)) = \int J^u \, d\mu^u_\beta < 0 \tag{2.3.2}$$

where

$$\chi_{\mu^u_\beta} = -\int J^u \, d\mu^u_\beta$$

is the Lyapunov exponent in the unstable direction, and

$$\left(\frac{\partial G^u}{\partial x}\right)(\beta, -F^u(\beta)) = \int \xi^u \, d\mu^u_\beta < 0 \tag{2.3.3}$$

Using (2.3.1), (2.3.2), and an inversion theorem, we prove that the function F^u (resp. F^s) is real analytic on $\mathbb{R}$; using (2.3.3), we prove that it is strictly increasing on $\mathbb{R}$, since we have for any real β

$$(F^u)'(\beta) = \frac{\int \xi^u \, d\mu^u_\beta}{\int J^u \, d\mu^u_\beta} > 0 \tag{2.3.4}$$

We have also the following property:

Theorem 2.3.1. The function F^u is either linear (degenerate case) (this is the case when J^u is homologous to $c\xi^u$) or strictly concave.

Proof of Theorem 2.3.1. The proof works in two steps and is much harder than the one in ref. 21.

- If there exists a constant c such that J^u is homologous to $c\xi^u$ ($c > 0$), then we have from (2.3.4) for any real β

$$(F^u)'(\beta) = c$$

and the function F^u is linear. In this case, its Legendre–Fenchel transform f^u [see (0.4)] is only defined at the point c with $f^u(c) = c$ [and $P(cJ^u) = 0$].

- Otherwise, we differentiate (2.3.1) in order to obtain

$$(F^u)''(\beta) = \left[\frac{\partial}{\partial x}\left(\frac{\partial G^u/\partial x}{\partial G^u/\partial y}\right) - \left(\frac{\partial G^u/\partial x}{\partial G^u/\partial y}\right)\frac{\partial}{\partial y}\left(\frac{\partial G^u/\partial x}{\partial G^u/\partial y}\right)\right](\beta, -F^u(\beta))$$

which becomes

$$(F^u)''(\beta) = \frac{(\partial^2 G^u/\partial x^2)(\partial G^u/\partial y)^2 - 2(\partial G^u/\partial x)(\partial G^u/\partial y)(\partial^2 G^u/\partial x\,\partial y) + (\partial^2 G^u/\partial y^2)(\partial G^u/\partial x)^2}{(\partial G^u/\partial y)^3}$$
$$(\beta, -F^u(\beta))$$

Using (2.3.2)–(2.3.4), we obtain then for any real β

$$(F^u)''(\beta) = \frac{(F^u)'(\beta)^2\,(\partial^2 G^u/\partial y^2) - 2(F^u)'(\beta)(\partial^2 G^u/\partial x\,\partial y) + (\partial^2 G^u/\partial x^2)}{(\partial G^u/\partial y)}$$
$$(\beta, -F^u(\beta)) \tag{2.3.5}$$

Following ref. 18, we identify

$$\left(\frac{\partial^2 G^u}{\partial x\,\partial y}\right)(\beta, -F^u(\beta)) = \sum_{k\in\mathbb{Z}} \left\{\int \xi^u \times (J^u \circ g^k)\,d\mu^u_\beta - \left(\int \xi^u\,d\mu^u_\beta\right) \times \left(\int J^u\,d\mu^u_\beta\right)\right\}$$

and for any real w [mod(2π)] we get

$$\sum_{k\in\mathbb{Z}} e^{-ikw} \left\{\int J^u \times (J^u \circ g^k)\,d\mu^u_\beta - \left(\int J^u\,d\mu^u_\beta\right)^2\right\} \geqslant 0$$

and

$$\sum_{k\in\mathbb{Z}} e^{-ikw} \left\{\int \xi^u \times (\xi^u \circ g^k)\,d\mu^u_\beta - \left(\int \xi^u\,d\mu^u_\beta\right)^2\right\} \geqslant 0$$

Since the property "there exists a constant L such that $d(x, [x; y]) \leqslant Ld(x, y)$" is satisfied,[18] the map

$$Q\colon\ C^\gamma(\Lambda) \to \mathbb{R}^+$$
$$h \to \sum_{k\in\mathbb{Z}} \left\{\int h \times (h \circ g^k)\,d\mu^u_\beta - \left(\int h\,d\mu^u_\beta\right)^2\right\}$$

(γ is such that the functions ξ^u and J^u are γ-Hölder) is a half-definite quadratic form on $C^\gamma(\Lambda)$, and its kernel is the set

$$E = \{C + K \circ g - K\colon c \in \mathbb{R},\ K \in C^\gamma(\Lambda)\}$$

If p represents the bilinear form associated to Q on $[C^\gamma(\Lambda)]^2$, we can write

$$\left(\frac{\partial^2 G^u}{\partial x^2}\right)(\beta, -F^u(\beta)) = Q(\xi^u)$$

$$\left(\frac{\partial^2 G^u}{\partial x\,\partial y}\right)(\beta, -F^u(\beta)) = p(\xi^u; J^u)$$

$$\left(\frac{\partial^2 G^u}{\partial y^2}\right)(\beta, -F^u(\beta)) = Q(J^u)$$

The expression (2.3.5) becomes

$$(F^u)''(\beta) = \frac{1}{\int J^u\, d\mu^u_\beta}\, Q(\xi^u - (F^u)'(\beta)\, J^u) \tag{2.3.6}$$

The map Q is positive on the vector field Vect(ξ^u; J^u) since the funtions ξ^u and J^u are not homologous. We have then for any real β

$$Q(\xi^u - (F^u)'(\beta)\, J^u) > 0$$

and using (2.3.2) and (2.3.6), we obtain for any real β

$$(F^u)''(\beta) < 0 \tag{2.3.7}$$

In that case the function f^u is defined on an interval $[\alpha^u_1; \alpha^u_2] \subset \mathbb{R}^{+*}$, is real analytic, and is strictly concave on $]\alpha^u_1; \alpha^u_2[$ with

$$(f^u)''(\alpha) = \frac{1}{(F^u)''\,[(f^u)'(\alpha)]} \tag{2.3.8}$$

We have similar results with F^s and f^s, and we prove that for any real $\alpha \in\,]\alpha_1; \alpha_2[$

$$f(\alpha) = f^u\{(F^u)'\,[f'(\alpha)]\} + f^s\{(F^s)'\,[f'(\alpha)]\} \tag{2.3.9}$$

Following ref. 21, we prove that we have for any real $\alpha \in\,]\alpha_1; \alpha_2[$

$$f(\alpha) = HD(C_\alpha) = HD(C^+_\alpha) = HD(C^-_\alpha) \tag{2.3.10}$$

The inequality $HD(C^\pm_\alpha) \leqslant f(\alpha)$ comes from a large-deviations result. The reverse inequality $HD(C^\pm_\alpha) \geqslant f(\alpha)$ uses Frostman's lemma.

We prove also that

$$\alpha_1 = \alpha^u_1 + \alpha^s_1 \qquad \text{and} \qquad \alpha_2 = \alpha^u_2 + \alpha^s_2$$

where we have, for instance,

$$\alpha_1^u = \inf_{\beta \in \mathbb{R}} (F^u)'(\beta) = \lim_{\beta \to +\infty} (F^u)'(\beta)$$

$$\alpha_2^u = \sup_{\beta \in \mathbb{R}} (F^u)'(\beta) = \lim_{\beta \to -\infty} (F^u)'(\beta)$$

and we can find g-invariant measures ρ_1 and ρ_2 such that

$$\alpha_1^u = \frac{\int \xi^u \, d\rho_1}{\int J^u \, d\rho_1} \quad \text{and} \quad \alpha_2^u = \frac{\int \xi^u \, d\rho_2}{\int J^u \, d\rho_2} \tag{2.3.11}$$

and we have also

$$f^u(\alpha_1^u) = \frac{h_{\rho_1}}{\int -J^u \, d\rho_1} \quad \text{and} \quad f^u(\alpha_2^u) = \frac{h_{\rho_2}}{\int -J^u \, d\rho_2} \tag{2.3.12}$$

We find therefore

$$f(\alpha_1) = f^u(\alpha_1^u) + f^s(\alpha_1^s) \quad \text{and} \quad f(\alpha_2) = f^u(\alpha_2^u) + f^s(\alpha_2^s) \tag{2.3.13}$$

and all the functions are defined at their boundaries. We identify now the values taken by the function f^u with

$$\text{for} \quad \alpha \in \,]\alpha_1; \alpha_2[\qquad \text{let} \quad \beta = (f^u)'(\alpha)$$

then we have[14, 15, 21]

$$f^u(\alpha) = HD(C_\alpha^u) = HD(\mu_\beta^u) \tag{2.3.14}$$

with $\mu_\beta^u(C_\alpha^u) = 1$ (C_α^u is the singularity set of μ^u), and

$$f^u(\alpha_1^u) = HD(\rho_1) \quad \text{and} \quad f^u(\alpha_2^u) = HD(\rho_2) \tag{2.3.15}$$

We use these results to obtain the main theorem.

3. PROOF OF THE DIMENSION SPECTRUM THEOREM

We generalize results obtained in ref. 17 in dimension one, and these results are similar to those in ref. 21. They are as follows.

Theorem 3.1. For any real $\alpha \in [\alpha_1; \alpha_2]$ there exists a g-invariant measure μ_α such that

$$f(\alpha) = HD(\mu_\alpha) \quad \text{and} \quad \frac{\operatorname{Log} \mu(R)}{\operatorname{Log} |R|} \xrightarrow[|R| \to 0]{} \alpha \quad \mu_\alpha \text{ a.e.}$$

Moreover, there exist positive reals τ and η such that

$$[C_\tau^u; C_\eta^s] \subset C_\alpha$$

and the Hausdorff dimensions of the two sets coincide.

It suffices to note:

- For $\alpha \in]\alpha_1; \alpha_2[$ (and $\beta = f'(\alpha)$), $\mu_\alpha = \mu_\beta^u \times \mu_\beta^s$.
- For $\alpha = \alpha_1$, $\mu_\alpha = \rho_1 \times \xi_1$.
- For $\alpha = \alpha_2$, $\mu = \rho_2 \times \xi_2$.

Here the measures ξ_1 and ξ_2 are associated to α_2^s and α_2^s [see (2.3.11) and (2.3.12)].

The function f is degenerate if and only if the functions f^u and f^s are degenerate. In that case, we have

$$c = c^u + c^s, \qquad \text{with} \quad f(c) = c, \qquad f^u(c^u) = c^u, \quad f^s(c^s) = c^s$$

If the function f takes the value 2, then the measure μ is absolutely continuous to the Lebesgue measure.[1]

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